Some really big numbers

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Problem

Given the following two sequences

$$P = 1, 2^2, 3^{3^3}, 4^{4^{4^4}}, ...$$

and

$$F = 1!, 2!!, 3!!!, 4!!!!, ...$$

which one grows quicker than the other?

1 Some preliminary comments and definitions

A number n with n factorials behind it is written as

$$n[!^n].$$

The factorial should be read from left to right. So for example $3[!]^3 = (((3!)!)!) = ((6!)!) = 720!$ A number written as n to the power n to the power n etc. is written as

$$n^{n^{\cdot^{\cdot^{n}}}}.$$

2 Some groundwork: a few corollaries

Before we go on to find out which of the sequences grows quicker, we will have to prove a few corollaries before concluding that F grows faster than P.

Corollary 1 *For every $n, a \in I\!N$ and $n > 3$ the following statement is true:*

$$n[!^{a-1}] > n^a.$$

Although it seems trivial it still takes a little effort to prove this corollary. First note that $n! > n^2$. Because $n > 3$ we can write $n! > n(n-1)(n-2) = n^3 - 3n^2 + 2n > n^2 + n^2(n-4) > n^2$. Now, it follows that if $n^{a-1}|n[!^a]$ then also $n^a|n[!^a]!$. This is true because $n[!^a]! = n[!^a](n[!^a]-1)...n(n-1)...3 \cdot 2 \cdot 1 = (n[!^a] \cdot n) \cdot ((n[!^a]-1)(n[!^a]-2)...(n+1)(n-1)...3 \cdot 2 \cdot 1)$, which is devisable by $n \cdot n^{a-1} = n^a$. So the following statement is true: if

$$n^{a-1}|n[!^a] \Rightarrow n^a|n[!^{a+1}]!$$

then

$$n^{a-1} < n[!^{a}] \Rightarrow n^{a} < n[!^{a+1}]!$$

By induction we have proved the corollary (the strict inequality doesn't really have to be proved).

Corollary 2 *For every $n, a \in I\!N$ with $n > 3$ and $a > 2$ then following statement is true:*

$$n[!^{a}] > n^{n[!^{a-1}]}.$$

This corollary involves a bit more work. Let us examine the following:

$$\log_n\left(n[!^{a}]\right) = \log_n\left(n[!^{a-1}]!\right) =$$

$$= \sum_{i=0}^{n[!^{a-1}]-1} \log_n\left(n[!^{a-1}] - i\right)$$

First note that all the terms of the summation bigger or equal than n are bigger than 1 and the last $n-1$ terms of the summation are all smaller than 1. Second, with the help of corollary (1), I can conclude the following: for every $j < n$ the following inequality is true

$$\log_n\left(n[!^{a-1}] - j\right) > \log_n\left(\frac{n[!^{a-1}]}{n}\right) > \log_n\left(n^{a}\right) - \log_n\left(n\right) = a - 1$$

So now we chop up the summation into smaller bits:

$$\sum_{i=0}^{n[!^{a-1}]-1} \log_n\left(n[!^{a-1}] - i\right) = \sum_{i=0}^{n-1} \log_n\left(n[!^{a-1}] - i\right) + ..$$

$$.. + \sum_{i=n}^{n[!^{a-1}]-n} \log_n\left(n[!^{n-1}] - i\right) + \sum_{j=1}^{n-1} \log_n\left(j\right) >$$

$$n(a-1) + \left(n[!^{a-1}] - n\right) - n$$

Note that $a > 2 \Rightarrow n(a-1) > 2n$. Therefore

$$n(a-1) + \left(n[!^{a-1}] - n\right) - n \geq n[!^{a-1}] = \log_n\left(n^{n[!^{a-1}]}\right).$$

So we can conclude:

$$\log_n\left(n[!^{a}]\right) > \log_n\left(n^{n[!^{a-1}]}\right).$$

And therefore:

$$n[!^{a}] > n^{n[!^{a-1}]}.$$

3 Conclusion

With this last corollary we can prove by concatenation that for every $n \in I\!N$ the following is true

$$n[!^{n}] > n^{n[!^{n-1}]} > n^{n^{n[!^{n-2}]}} > ... > n^{n^{\cdot^{\cdot^{n[!^{2}]}}}} > n^{n^{\cdot^{\cdot^{n^{n^{n!}}}}}} > n^{n^{\cdot^{\cdot^{n}}}}.$$

Since every term of the sequence F is bigger than P, the sequence F can be said to grow faster than P.